# Geometrical Methods in Stringed Keyboard Instrument Design and Construction

**Stephen Birkett and William Jurgenson**

## Summary

It is generally believed that historical stringed keyboard instruments were constructed by transferring their pre-determined shape and dimensions from a master reference design of some sort. This transfer would have had to be accomplished by a measurement process, or, in general, using some technique that compared two dimensions (messen heist vergleichen). If this approach was indeed used we are forced to conclude that: (1) some method existed for developing the original reference design; (2) a mechanism was needed for recording and storing the reference design; and (3) the transfer of dimensions could be accomplished with adequate and repeatable accuracy. The conventional organological solution is that instrument makers recorded their designs, and physically transferred them to an instrument under construction, using reference "sticks", or diapasons. However this leaves the most important question unanswered, viz. how was the original design developed? We propose that a completely different approach was used, solving all three aspects of instrument design and construction in a simple pragmatic method of working. In particular, there is extensive evidence for the ubiquitous use of constructive proportional geometry, not only to develop the original design in accordance with acoustical, practical and aesthetic requirements, but also to record, retrieve and transfer this to an instrument under construction. In this way it is possible to dispense altogether with measurement in the process of achieving a repeatable, and highly self-consistent and accurate case design, with the exception of a single reference dimension from which the entire case design can be derived. Case dimensions have been examined from the earliest extant keyboard instruments to mid-nineteenth century pianos and found to be consistent with the proposed method of working. This relies only on simple and fundamental practical knowledge that was well-known, and commonly employed in related crafts. This article discusses the specific case layout schemes for five-octave Viennese grand pianos from 1780 to about 1800, and illustrates these by presenting reconstructions of two different methods used by J.A Stein.

## *The Role of Geometry in the Crafts*

The earliest writers classified mathematics, in particular geometry, into two types, practical and theoretical. As early as the thirteenth century, a third, and quite distinct, variety of geometry began to be recognized, viz constructive geometry, commonly referred to as *geometria fabrorum*. Geometrical forms played a central role as devices for the crafts, such as masonry and carpentry. Physical manipulation of these forms, using simple instruments and tools, circumvented the need for a craftsman to perform any algebraic calculations in design and fabrication. Since this constructive geometry was taught in an oral tradition from memory and experience, using rules-of-thumb, the precise techniques used in the various crafts are difficult to find documented. However it is feasible to reconstruct the geometric design principles relevant to a particular extant artifact, by analysing it in the context of constructive geometry as it would have been practised by the contemporary craftsman who made it.

“Every art has its own materials, forms and measures...it arises out of the fundamental basis of geometry through the manipulation of compasses and how it should be brought into the correct proportions. " [1] Clearly proportion played a central role, but it is important not to put the cart before the horse. The correct causality, as explained by Roriczer, implies proportions are derived as an implicit consequence of the intrinsic practical geometry used, rather than being explicitly incorporated into a design. Reporting the "observed" proportions in an extant artifact is meaningless from a practical point of view, and says nothing about the working practices of the craftsman who made it, unless the essential links to practical shop geometry are determined. Unfortunately much of the published literature on design[2] relates proportion only to observation, and fails to make the link to shop practice and what was actually done. "Man has written infinitely about proportions. Every year books appear about new triangles, rectangles, polygons, golden or otherwise and keys and rules trying to solve the architectural secrets of the ancients."[3] This approach is simply a study of "geometrical shapes rather than in architectural works."[4] The same comments apply to most of the published proportional analyses of musical instruments.

A fundamental element in practical constructive geometry in the crafts is the concept of *alte Shuhe*, as described, for instance, by the sixteenth century master mason Lorenz Lechler[5]. This single modular unit forms the basis of all subsequent design. It is important to realize that the modular unit was selected for convenience or practicality, and there is no *a priori* reason to expect it to agree with any particular local unit of measurement. Lechler makes this quite clear in his instructions to "take the wall thickness of the choir, whether it be large or small" and derive subsequent dimensions from this modular unit. This emphasizes the practical utility of the concept of alte Shuhe – the modular unit provided the continuity between successive generations of workers as a large project was completed over many decades.

In instrument design the modular unit serves the same purpose, in principle, being derived from practical considerations in relation to the mechanical functioning, nothing more. It is erroneous to make any conclusions based on theoretical or historical assumptions about any direct relation between a local unit of measure (foot etc.) and the modular unit used by craftsmen such as instrument builders. In his instructions for laying out a clavisimbalum (harpsichord), Arnaut de Zwolle[6], for instance, indicates the modular unit in harpsichord layout "can be chosen at the convenience of the builder". Koster's conclusion[7], based on the linguistic relation between "alte Shuhe" and "Fuss", that "Schmuttermayer's[8] old shoe was presumably just a standard foot according to the local unit of measure" is inconsistent with evidence from historical sources. Theories developed based on this major assumption will naturally lead one far astray from the actual methods used by early craftsmen.

Constructive geometry was used in a spirit of flexible pragmatism, and different masters in a particular craft were free to adapt a "traditional" design to suit their own purposes, if they so chose. No knowledge of theoretical relationships was required to derive a new design, nor to implement a known one which was learned as an apprentice. These techniques are analogous to using recipes in cooking, consisting of a series of specific steps.

---

[1] Mathes Roriczer, Geometria Deutsch, Regensburg, 1488. Fascimile ed. Ferdinand Geldner, Wiesbaden, 1965.

[2] For instance: Kevin Coates, Geometry, Proportion and the Art of Lutherie, Oxford University Press, 1985.

[3] Ernst Neufert, Bauordnungslehre, 1943. Quoted as footnote 19 in M. Borissavlievitch, The Golden Number and the Scientific Aesthetics of Architecture, Alec Tiranti, London 1958.

[4] Ibid., p. 28.

[5] Lorenz Lechler, Unterweisung, 1516. See discussion in: Lon Shelby, The geometrical knowledge of the master masons, Speculum 47 (1972): p 395-421.

[6] Arnaut de Zwolle, Fifteenth Century Burgundian manuscript. See Stewart Pollens, The Early Pianoforte, Cambridge University Press, 1995. Chap. 1 contains a complete translation of Arnaut's instructions for constructing a clavisimbalum and other stringed keyboard instruments.

[7] John Koster, Toward the reconstruction of the Ruckers' geometrical methods, Halle 1998.

[8] Hans Schmuttermayer, a medieval craftsman writer, 1488.

A variation of the methods based on constructive geometry, known to have been practised by masons, and apparently used by some instrument builders, uses a numerical sequence of multiples of the modular unit dimension to derive the appropriate proportions. For instance ratios of successive pairs of dimensions in the sequence 1, 2, 3, 5, 8, 13, 21 etc. give an approximation of the golden mean. Although this is the well-known sequence of Fibonacci numbers, it would be incorrect to suppose the early builder had any knowledge of this mathematics, nor did they need this to use these dimensions appropriately. Evidence for the use of this sequence as the basis for proportional lengths derived from a modular unit is contained explicitly in the instructions for designing a harpsichord, printed by Arnaut de Zwolle in about 1430[9]. The approach based on numerical sequences will not be discussed further here.

The most common proportions that form the basis of many useful geometric constructions are $\sqrt{2}$ and $\sqrt{5}$, and the golden mean $\phi$ (= 0.618...), as well as the obvious integral multiples 2, 3, 4 and so on. All of these are ubiquitous in keyboard instrument designs. Figure 1 shows a well-known geometric construction for $\phi$ based on a right triangle ACB with sides 1, 2 and $\sqrt{5}$. Side AC is transferred with the compass to the hypotenuse AB, giving the dimension $\sqrt{5} - 1 = 2\phi$. With the compass centred at B, the $\sqrt{5} - 1$ is transferred to the side CB. This divides side CB into two segments with lengths in the golden mean ratio, so that CD:DB = DB:CB = $\phi$ i.e. the ratio of the length of the shorter segment (the *minor*) to the longer segment (the *major*), is the same as the ratio of the length of the major segment to that of the whole side. Although this simple construction illustrates an important geometric property of the golden mean, as will be explained, it is unlikely to have been useful to the stringed keyboard instrument maker.

The tools that are required for geometric constructions are few and simple. Early writers describe the basic tools as the compasses, the large beam compass (trammel) which is particularly useful in harpsichord and piano designs, the straight-edge, and the ruler. Secondary tools such as the square are useful but not essential, as perpendicular lines are easily constructed geometrically. Layout tools are sized in relation to the artifact being constructed. Roubo[10] illustrates the architectural carpenters' workshop, where enormous doors and panelling are constructed. In the corner of the workshop a huge trammel is clearly visible in the illustration, its size commensurate with the size of the objects being laid out. For fluegel-shaped keyboard instruments we would expect a trammel of length about 2 m to be available. This list of simple tools required for instrument building is confirmed in Dom Bedos' work on organ building[11].

## *Geometry in Stringed Keyboard Instrument Design*

Hubbard's comments on harpsichord design and construction are astute: "It is remarkable that old makers do not seem to have worked very much from drawings...In all our inventories there are only three hints of drawings or templates, and none is very specific...I have seen one Italian harpsichord in which the maker had drawn the plan view of the instrument in full scale on the inside of the bottom. Like a beaver building his dam, then, the maker constructed his case, guided by experience for the length and by the known size of a keyboard of the projected range for the width." [12]

There are three conceptually distinct aspects to the problem: (i) initial design; (ii) data storage; and, (iii) data retrieval. First the basic shape and dimensions of the instrument, and the location

---

[9] Arnaut de Zwolle, ref. Pollens, Op. Cit.

[10] Roubo, L'Art du Menuisier, 1769.

[11] Dom Bedos, L'Art du Facteur d'Orgues, 1776.

[12] Frank Hubbard, Three Centuries of Harpsichord Making, Harvard University Press, Cambridge, Mass., 1965.

of main components, must be determined, and these must be consistent with the projected keyboard size, and the action and stringband. Furthermore the basic design must be mechanically, acoustically and aesthetically sound. Second, the basic design parameters must be stored by some means. Modern theory has proposed the use of a reference drawing for this purpose, although there seems to be negligible supporting evidence. Alternatively it is often suggested that important dimensions were recorded on a reference "diapason" stick. Third, the basic design must be retrieved and transferred to an instrument under construction. The conventional organological solution is that a reference diapason stick was physically used to transfer dimensions to an instrument, although there are a number of reasons why this would not have been practicable in a busy shop situation. Most important, the diapason stick method leaves unanswered the most important question of how the original design itself was developed.

Modern engineering design methods separate the three tasks described above: the basic design is developed via various ad hoc, or scientific principles; this is recorded on a physical, or computer, reference drawing; the transfer of all points in the construction process is accomplished via measurement, whereby each point is located by its coordinates in a "reference grid" on the object being constructed. This modern manufacturing methodology was used for stringed keyboard instruments toward the latter part of the Nineteenth Century, as confirmed, for instance, by Bluethner's detailed description of piano design principles in his "Lehrbuch"[13]. There is an inherent limitation to the accuracy of locating points via measuring their coordinates, even with the most sophisticated modern techniques. For some modern piano manufacturers, inaccurate point location has become a serious manufacturing issue, and they have recently attempted to ameliorate the situation through the use of computer-aided manufacturing technology.

Consistency and accuracy are important, of course, but, in particular, it is the internal, or self, consistency which is critical to the correct functioning and assembly of a stringed keyboard instrument. Practical constructive geometry can be used as a very effective tool to solve simultaneously all three of the design/construction issues described above. *This approach is fundamentally different from modern thinking.* It is also simpler, and more reliable, then the modern one, especially for the individual craftsman working in a small shop setting. The basic design consists of a series of geometrical constructions, which would have been learned orally, passed on from master to apprentice. These constructions could be performed repeatedly any number of times, with consistently predictable results (as seen, for instance, in the remarkable consistency in some of the dimensions observed on Ruckers harpsichords built over a period of some 150 years). In particular there is no need to record the design, nor is it necessary to transfer any dimensions from a reference design to an instrument under construction. The geometry is simply reconstructed each time a new instrument is begun. Using geometry rather than coordinate measurements to locate points has other important advantages: since all dimensions are derived directly from other dimensions previously constructed, internal consistency is automatically achieved at a very high level; geometric constructions are, in general, a more accurate means to locate points, as compared to measurement; no inaccuracy is introduced through the transfer of reference dimensions to an instrument; and, it is very simple to scale a design larger or smaller, for instance, to accommodate variation in required keyboard compass.

## *The Modular Unit for Stringed Keyboard Instruments*

Geometric constructions use proportional relations to determine new dimensions from old ones. To define the absolute size of an instrument implies that one specific dimension, derived from the modular unit, must be used as the starting point. For fleugel-shaped keyboard instruments, such

---

[13] J. Bluethner and Gretschel, Lehrbuch des Pianofortebaues in seiner Geschichte, Theorie und Technik, Weimar, 1886 (second edition). Chap. 16: Der Bau des Fluegels.

as harpsichords and grand pianos, this modular unit is related to the stringband width, and for rectangular instruments, such as clavichords and virginals, the keywell width. Only fluegel-shaped[14] stringed keyboard instruments will be considered in this article, and the detailed emphasis concentrates specifically on the design of five-octave Viennese pianos.

Stringband spacing – i.e. the distance between the strings - (almost) invariably seems to have been designed on the basis of an allowance of one half-inch per string group (choir), using some local unit of measurement[15]. The original half-inch measure used by a particular builder can quite easily be discovered, and used in subsequent analysis, by constructing a ruler based on the stringband spacing in an extant instrument. Additional confidence in the proposed half-inch unit can be derived by considering various arbitrary dimensions on the instrument, which are very often strongly related to multiples of half-inch, or twelths of one inch (the convenient historical inch divisions called "lines"). The half-inch measure is the fundamental basis for all that follows, but it is not the modular unit for the geometric construction. Neither is the foot the modular unit, however many inches it may have (12 or 11 etc), and, in fact the foot appears to not be relevant to stringed keyboard instrument design at all, despite the great importance attached to it by some authors[16].

Stringband spacing, together with the known keyboard compass, determined the overall width of the stringband. This stringband width then determines the required width of the keyboard panel at the strike, or plucking, point. The keyboard panel generally began as a key sheet of width similar to that required at the back. Almost always, though, an arbitrary half-inch multiple (sometimes quarter-inch) seems to have been chosen for the size of this panel[17]. The keysheet must be accommodated in the keyframe which holds it, and this will include key cheeks (possibly of zero width, as in Ruckers). The early builder arrived at the modular unit by allowing for the known key panel width, the two key cheeks, an allowance for panel/cheek clearance, plus the two case walls of the instrument. This dimension, which we will denote by 2W, therefore determines the overall outside width of the inner case. By this is meant the width of the basic structure as laid out on the bottom boards on the bench at the start of construction, disregarding any exterior case "skin" which may have been added later in the construction. *The half-width dimension, W, is the fundamental modular unit from which all other remaining dimensions in the instrument are derived.* The half-width also defines the centreline of the instrument case, an important concept in the historical approach to geometrical design.

To illustrate these concepts, a 1783 piano by J.A. Stein (Wuerttembergisches Landesmusuem, Stuttgart) has a keypanel of width 31 inches, using the Stein inch of 26 mm (corresponding to a foot of 312 mm), which we have determined from the stringband spacing of several extant Stein pianos[18]. The two inner case liners and the two key cheeks each are 1 inch wide. A three-quarter

---

[14] The wing-shaped keyboard instrument with strings horizontal to the floor, such as for harpsichords, grand pianos etc. i.e. not rectangular, and not upright.

[15] W. Jurgenson, The Whole Truth?, in press.

[16] For instance: Grant O'Brien, Ruckers: A Harpsichord and Virginal Building Tradition, Cambridge University Press, 1990

[17] Jurgenson, Op. Cit.

[18] Note the significant difference between the Stein foot of 312 mm and the "standard" Viennese foot of 316 mm for this period. It is interesting that, as determined by the authors, Nanette Streicher, Stein's daughter, continued to use the "Stein" 312 mm foot until her retirement from piano building in the late 1820s, despite having moved to Vienna at the turn of the Nineteenth Century i.e. she did not adopt the standard local foot used by contemporary Viennese builders. Small differences in the foot (and therefore the inch) used by different builders have major historical implications for the organological identification of instruments, for the deduction of working methods and practices of makers of stringed keyboard instruments, and for the analysis of the historical relationships between different builders. Traditional organological methods for determining the foot used by a particular builder are not satisfactory, and lead to

inch allowance is given for clearance. Thus the width of the case is 31+2+2+3/4 = 35 3/4 inch. These dimensions are consistent with those observed on the instrument, and with the overall inner case width, 2W, which measures 929 mm. For such instruments as Viennese pianos, the width 2W is also the width of the bottom boards at the gap. Since the outer case is simply a "skin" that is added later, it does not enter into the basic design which is constructed on the bottom boards.

By deriving the modular design unit, the half-width W, and the full-width 2W, from the stringband, of known spacing and size, and the keysheet, with allowance for keycheeks, clearance, and case walls, the instrument builder automatically guaranteed that the mechanism, structure, and stringband would fit in the case once it was constructed. No further consideration of this was required. Design layout proceeded on a bottom blank of suitable width to accommodate the 2W dimension. A panel of boards of adequate length was assembled by edge-gluing sufficient boards to achieve the required width. The exact length was not important at this point, provided the panel was everywhere longer than necessary – this could easily be estimated by rough comparison to another instrument in the shop, or to a sketch on the benchtop, quickly constructed using the geometry.[19]

## *Constructing the Long Dimensions and Outline Plan*

The required longitudinal dimensions, and the outline of the plan shape, were derived by geometrical construction directly from the modular dimension W, which was first marked on the blank for the bottom boards. Examination of many extant stringed keyboard instruments makes it apparent that two distinct basic approaches were used. In the first method, which applies to Italian harpsichords, or Florentine pianos, the position of the bridge was located, and in many cases actually drawn, directly on the bottom boards. The curve of the bentside was then drawn on the bottom by translating the bridge curve a specified distance outward (half-inch multiple). The principle of this method was described as early as 1430 in the instructions for laying out a harpsichord by Arnaut de Zwolle[20]. It applies particularly to instruments in which Pythagorean scaling proceeds deeply into the bass section of the stringband, and for which there is generally no straight section of bentside. For such instruments a simple geometric method was used to construct the tail, and therefore determine the overall length. This type of instrument geometry will not be discussed further here.

The second type of instrument may be characterized as an holistic design, for which the entire case was laid out without explicitly considering the string lengths at all, i.e. with an "outside/in" approach. The geometry was developed in such a way that the desired scaling of the stringband could be accommodated in the case, and this was achieved automatically. For such instruments, Pythagorean scaling was not continued deep into the tenor, and, since the lower string lengths are quite arbitrary, a straight section of bentside was employed. The bridge will subsequently be parallel to all, or most, of this straight bentside section. This type of construction will be explained in the remainder of the article, which focuses on five-octave Viennese pianos from 1780 to 1800.

---

erroneous and inconsistent conclusions. The geometric working methods proposed in this article imply the stringband spacing will always be a completely reliable technique for deriving the builder's foot (and inch) measure, since it is the basis for the geometry of the entire instrument, and the only important dimension that was actually obtained by direct measurement.

[19] The grain of the bottom boards on a Stein piano runs parallel to the long bridge, i.e. at an angle to the spine edge, therefore he probably used such a benchtop sketch as an aid to gluing up the bottom panel.

[20] See Pollens, Op. Cit.

## *The Constructive Geometry of Viennese Pianos*

The basic design of all Viennese pianos prior to the mid-Nineteenth Century can be derived using several simple shop constructions, although this is far from obvious based on superficial measurement of extant pianos. Even when it is possible to gain access to the inside bottom surface, a builder's marks on the bottom cannot be expected to be still visible, given the age of these pianos. In any case, most of these marks would have been obscured by the subsequent construction, as parts were glued onto the bottom boards. Therefore, in order to deduce conclusively that a particular geometric construction has indeed been used in the design of an extant instrument, a very small tolerance must be permitted, in comparing observed and predicted dimensions[21]. To reflect this, all such comparisons in the geometrical analyses reported here are based on a tolerance of only 1 mm i.e. if the difference for specific dimensions that are proposed to have been located via direct geometric construction is more than 1 mm, the theory is rejected. This necessitates the collection of extremely accurate data from an instrument, a process which is very time-consuming. High-quality technical drawings have been consulted as a data source when these are available, and accurate enough, as well as personal observations of some instruments. For some pianos, such as the Duelcken in the Smithsonian Institute, Washington, some case distortion must be "removed" before the original geometry can be accurately reconstructed. Therefore, if technical drawings are used, the distortion must be unambiguously indicated on the drawing (it is on the Smithsonian Duelcken). Despite these difficulties, once the original geometry has been reconstructed from the extant evidence, and compared to the proposed method, there is little doubt when they are in agreement. The agreement between theoretical and observed dimensions is simply too close to be coincidence.

The proposed method of working described below is based on implied geometric constructions which become evident through examination, and subsequent analysis, of extant instruments. The only technical assumption made of the historical builder is a knowledge of well-known traditional geometrical shop constructions, which were universally practised by contemporary craftsmen. No understanding of the theory behind this geometry is assumed. With these reasonable assumptions, a remarkably simple, effective, accurate and consistent layout procedure is revealed.

Once the modular unit had been decided on, a long straight edge was planed on the blank for the bottom panel to define the spine side of the bottom. A *main reference line* was then constructed perpendicular to the spine edge, and intersecting it where the back of the gap was to be located. This line generally relates to the position of the bellyrail in some way, but the exact location varies between, and even within, the output of different piano builders[22]. In the process of

---

[21] This small tolerance is a requirement only for our comparison of predicted and observed dimensions. The original choice of 2W by the builder need not be especially accurate. Certainly minor random variation in this dimension was of no consequence, resulting in pianos from the same builder that may have had very slightly different widths, perhaps up to a few millimeters typically. However, once 2W was determined and marked on the bottom boards, all subsequent dimensions, located by geometric construction, were necessarily highly consistent with the actual 2W dimension used for that instrument. Therefore correct analysis demands a high degree of internal consistency. Our very small permitted deviation in comparing predicted and observed dimensions does not allow, for instance, that the bottom boards may have changed width between marking them and the time they were subsequently constrained by being glued to the framing (a situation analogous to changing the "aspect ratio" of a drawing, which causes the ratios of longitudinal to transverse dimensions to change). Such situations cannot be distinguished from erroneous theory. Therefore we have chosen to err on the side of allowing only the most pristine data to be accepted, discarding any that may be suspect, for any reason.

[22] A skewed bellyrail (e.g. Walter, Ruckers, etc) was located at one end with respect to the main reference line. The other end, and therefore indirectly the skew angle, was generally obtained as an inch multiple from the reference line. These facts are obvious once the main reference line in an extant instrument is located.

analysing an extant instrument, the identification of this main reference line is a key step to reconstructing the geometry. If this line were actually scribed on the bottom, it would probably subsequently have been obscured, because it so often was situated under the bellyrail, and this was usually glued to the bottom boards. However it is not necessary to actually scribe the reference line all the way across the bottom, so one can expect that this was probably not done anyway. All that is required are the two end points where the main reference line intersects the spine and cheek, and the intersection with the centreline i.e. three points that could be marked with small ticks. In particular, beginning at spine end of this main reference line, the builder simply set his compasses to the modular dimension W and marked points at distance W and 2W , along a perpendicular to the spine, to define the position of the centreline and the cheek line from the spine edge. The cheek line would then have been scribed parallel to the spine edge. This simple construction defines the keywell, except for its front edge, and locates the gap and bellyrail. The front depth of the keywell usually does not form part of the layout geometry and would have been determined later, to accommodate the anticipated key lengths.

The golden mean $\phi$ is central to the layout method for Viennese pianos. Many interesting and unique numerical properties of this number are well-known, but these are irrelevant to the needs of piano builders. Only the special geometric relationships that are present in figures with proportions related to the golden mean are required. A simple shop construction for the golden mean was described above and illustrated in Figure 1. However this particular construction is *not* useful to the piano builder, because all required layout points, derived from the modular unit W, must be constructed on the spine, centre, cheek, and main reference line, forming a rectangular grid. The intermediate step in the construction shown in Figure 1 results in the golden mean being placed on the hypotenuse of the triangle, i.e. a point not in the rectangular grid.

The *golden rectangle*, a rectangle with sides in the ratio 1 to $\phi$, is a geometrical figure of fundamental importance to the instrument builder. The rectangles FBCG and EADH, illustrated in Figure 2, are golden rectangles. A larger golden rectangle can be formed from a smaller one, by attaching a square to the long side of the smaller golden rectangle. Doing this forms a rectangle with sides in ratio $1+\phi$ to 1, which is the same as 1 to $\phi$[23]. Any golden rectangle, such as the shaded rectangle ABCD shown in Figure 2, can be decomposed in this way into a square AFGD and a smaller golden rectangle FBCG. Attaching another of the smaller golden rectangles on the other side of the square, EADH in the figure, results in a long rectangle with sides of length $\sqrt{5}$ and 1. ***The interlocked pair of golden rectangles - ABCD, shown shaded in the figure, and EFGH - joined by their common central square AFGD, is the basis for the shop construction that defines the skeleton of (almost) every five-octave Viennese piano***[24] .

The proportions of $\sqrt{5}$ and $\phi$ are not only related by the obvious numerical definition of the golden mean, $\phi = (\sqrt{5} - 1) / 2$, but also by the fundamental geometry illustrated in Figure 2. This relationship can be reversed, and the interlocked pair of golden rectangles easily constructed in their correct orientation inside any rectangle with sides of length $\sqrt{5}$ by 1. To do this the $\sqrt{5}$ side is bisected and a semicircle of radius $\sqrt{5} / 2$, centred at the midpoint, is drawn as shown. This semicircle intersects the opposite $\sqrt{5}$ side at the points A and F. These points are the corners of the square AFGD that defines the required interlocked golden rectangles. In the application of this geometry to five-octave Viennese pianos, point A is one of the key points that locate the bentside.

The *golden triangle* is a triangle with sides in the ratios $1 : \sqrt{\phi} : \phi$ (approximately 1 : 0.786 : 0.618). It can be easily constructed by transferring the long side of any golden rectangle to the

---

[23] Note that the number $\phi$ possesses the unique property that $1/\phi = 1+\phi$ .

[24] The five-octave pianos of Anton Walter are an exception to this, but only in detail.

opposite side, as shown in Figure 3. The golden triangle, and related $\sqrt{\phi}$ proportion, are historically important: the triangle, for example, forms the geometrical basis for the Great Pyramid at Giza; the successive drawer heights in many schools of furniture making are often related in $\sqrt{\phi}$ proportions; and, as we shall demonstrate, the golden triangle played a key role in the geometry of five-octave Viennese pianos.

It is important to realize that the constructions and geometry described above are not complex, although they may seem so to describe them formally. A shop apprentice, with no mathematical training at all, would have been able to learn them in a few minutes. No understanding of the theory behind them is required to use them as practical geometric constructions in the shop, nor to take advantage of the important geometric relationships they represent. This is the fundamental basis for constructive geometry. It is well-known that this geometric approach to design layout was traditional shop practice in the crafts, architectural work, and so on, therefore it can be expected that any piano builder would have had the required skills.

## *The Layout Procedure for Five-Octave Viennese Pianos*

A simple layout procedure, based on the constructive geometry described above, appears to have been almost universally used by builders of five-octave Viennese pianos. The main principle relies on defining the bentside position by constructing the four points shown in Figure 4, in relation to the rectangular grid of spine, centre, cheek and main reference line already described. Points B1 and B2 determine the line of the straight section of the bentside, both its angle and its location with respect to the spine and main reference line. Point B3 determines the end of the straight section and the starting point for the tail section, whether this is to be curved (J.A. Stein, 1783), angled (Anton Walter, 1800) or straight (Nanette Streicher, 1814). Point B4 locates the beginning of the curved section of the bentside. Additional points were needed for defining the treble bentside curve (and where this intersects the cheek line), and for the tail geometry, which also defines the overall length of the piano. Since these additional points varied somewhat between builders we will concentrate here on the location of the four bentside points B1, B2, B3, and B4, which all builders of Viennese pianos required.

Application of the geometry shown in Figure 2, scaled up by the modular unit size W, is illustrated in Figure 5 (the keyboard is to the right and the spine is at the bottom of the drawing along the side EC). Points A, B and C in Figure 5, where the main reference line intersects the cheek, centre and spine line respectively, are important points in the subsequent constructions. In describing the various constructions these three points will be referred to as the *cheek*, *centre* and *spine reference points* respectively. A rectangle of sides $W\sqrt{5}$ by W was first constructed, with one long side EC aligned with the spine edge, and one short side BC aligned with the main reference line at the gap. To produce the $W\sqrt{5}$ dimension in the correct position on the spine, the compasses, set to the modular unit W, were first placed at point A and the W dimension transferred to the cheek at point D shown. The trammel was then set between points C and D, and the $W\sqrt{5}$ dimension was swung down to the required position CE on the spine. No part of this procedure requires any measurement, nor is it necessary to know anything about the numerical dimensions. Only the proportional geometric relationships are important.

The $W\sqrt{5}$ dimension EC was then bisected with the compasses to give point F. With the compasses centred at F, the $W\sqrt{5}/2$ dimension FE was swung up to meet the centre line at point G shown. This <u>point G is the required first point which locates the bentside – point B1of Figure 4</u>. The distance from the bentside point B1 to the main reference line is then in a fixed proportion $(1+\phi)$ W in relation to the modular unit. The observed distance along the centre line from the bentside to the main reference line is consistently observed on extant five-octave pianos to be precisely in agreement with this theoretical dimension within the very close tolerance of 1 mm specified above. This proportional relationship is true regardless of variation in the width of the piano.

The second required bentside point B2 is located using the golden triangle construction shown in Figure 3. With the compass set between points H and G in Figure 5, the W dimension GH is swung down to the opposite side of the golden rectangle GHEJ, to define the point K. In practice it is only necessary to construct a perpendicular to the spine at point E, and find the point K where the compass arc intersects it. The triangle KHE is a golden triangle, and KE is dimension $W\sqrt{\phi}$. Point K in Figure 5 is the second required bentside point B2 of Figure 4.

Points B1 and B2 are sufficiently far apart that a straight line could reasonably be scribed through them to define the straight part of the bentside. On this line the other two bentside points, B3 and B4, were constructed by similar geometric methods, as was the point where the tail and spine intersect. There is some variation between different builders on the construction of these other points. An example is given as part of the constructions described in the following section.

The two points B1 and B2 uniquely define the straight part of the bentside, both the angle and the position in relation to the main reference line at the gap. The angle of the line through B1 and B2 has a tangent of $(1 - \sqrt{\phi}) / \phi = 0.3461$, corresponding to an angle of 19.09 degrees, precisely the observed angle formed by the bentside of five-octave Viennese pianos by Stein, Hoffmann, Langerer, Schiedmayer, Duelcken, Geschwister Stein, and many others. The universally consistent characteristic 19 degree bentside angle of most five-octave Viennese pianos is a direct consequence of the general use of this construction. It is highly unlikely that both the consistent proportional relation between width at the gap and the distance to the bentside, and the consistent 19 degree bentside angle, could have been the consequence of any other means than through the geometric construction described here.

## *Geometric Constructions for Pianos by J.A. Stein*

To illustrate the geometric layout methods, instructions are given for the re-construction of the methods required for two slightly different piano designs of J.A. Stein, a “Phase II” and “Phase III” piano[25]. The geometry for the Phase II design has been derived from a plan view line drawing of the 1783 Stein in the Boston Museum of Fine Arts[26], and guided by consideration of constructions derived for other similar five-octave Viennese pianos. The Phase III re-construction is based on extensive measurements taken on the 1783 Stein in the Wuerttembergisches Landesmuseum, Stuttgart[27]. The observed half-width modular dimension W for both pianos is 465 mm. The inner liner and framing is shown shaded in the diagrams of Figures 6 and 7. The bottom boards extend to the outside edges of the shaded liners, where all the construction points are located.

### Phase II Piano

The constructive geometry of a Stein Phase II piano is illustrated in Figure 6. The main reference line in this instance defines the front edge of the bellyrail, which was not skewed, and therefore exactly follows the reference line. The half-width 465 mm was first transferred to the cheek with

---

[25] Terminology of Michael Latcham, The Pianos of J.A. Stein, Haagsgemeentemuseum, 1993.

[26] John Koster, Keyboard Musical Instruments in the Boston Museum of Fine Arts, Museum of Fine Arts, Boston, 1997

[27] Latcham Op. Cit. dates the Stuttgart Stein as 1788 without any explanation, yet it is clearly dated 1783 in pencil inside. The internal design particulars indicate this piano is an example of Stein’s “Phase III” output, using the terminology in Latcham. Thus the Phase II and Phase III pianos presented here are both from the same year 1783.

the compasses. This cheek point also defines the front edge of the keywell in this case. Its main function, of course, is for constructing the $465\sqrt{5}$ dimension (1040 mm), which was transferred with the trammel to the spine as shown (the 465 + 575 mm on the diagram). The 1040 mm dimension was then bisected on the spine with the compasses, and an arc of radius 520 mm was swung to locate the first bentside point B1 on the centre line. Point B1 is at a distance $465(1+\phi) = 752$ mm from the centre reference point.

To locate the second bentside point B2, the trammel was first set between B1 and the centre reference point (752 mm). This dimension was transferred to the spine and marked off from the spine reference point. A perpendicular to the spine was constructed at the 1040 mm mark. With the compasses set to the 465 mm dimension between spine and centre line, and placed at the 752 mm mark on the spine, an arc was swung to meet the "1040 mm" perpendicular and define point B2.

The tail geometry, including the third required bentside point B3, where the tail begins, was constructed as follows. First the 465 mm W dimension was transferred to the spine from the main reference line as shown, with the compasses placed between the spine and centre reference points. The difference between the already-constructed 1040 mm and this 465 mm dimension is 575 mm (which also happens to be $2W\phi = 930\phi$ mm). The compasses were set to this 575 mm dimension and an arc, centred at the 1040 mm mark on the spine, was swung around to mark the point of the spine where the tail intersects it (i.e. this defines the precise length of the inner case where the bottom boards end). The rear 575 mm dimension was then bisected to give 287 mm from the tail point, and a perpendicular drawn from the spine to locate the point B3 where the bentside finishes and the tail begins.

Point B4, where the curve of the bentside begins, was located on the bentside line by first placing the trammel between the spine reference point and the 752 mm mark on the spine. A 752 mm arc centred at the spine reference point was swung to meet the bentside line, which intersection defines point B4.

The overall predicted length of the bottom boards calculated from the construction described above is 575 + 575 + 465 + 465 = 2080 mm.

### Phase III Piano

For the Stein Phase III piano, the construction began with a main reference line that corresponds to the *back* edge of the bellyrail. As for the Phase II piano, the bellyrail is not skewed, but it is somewhat thinner in the Phase III piano. In this construction, the 465 mm half-width dimension, transferred to the cheek line, does not define the front edge of the keywell, which was determined by another means. Since the bellyrail was now in *front* of the reference line it is "inside" the keywell area, effectively reducing the available space. Since the keys are the same length as those of a Phase II piano, such a keywell would not have been deep enough to accommodate them.

The $W\sqrt{5}$ rectangle (1040 mm by 465 mm), and bentside points B1, B2, and B4, are constructed as for the Phase II piano. Further confirmation of the use of this geometry is to be found in the position of the rear cross brace, the front edge of which lies precisely 752 mm behind the main reference line. It is very unlikely that this obscure position would have been used otherwise.

The tail geometry is slightly different in the Phase III Stein piano. A simple one-step method gave the location of the tail on the spine. The compasses were placed between the 1040 mm mark on the spine and point B1 (dimension is 547 mm). An arc centred at the 1040 mm mark was swung around to intersect the spine and define the tail point. Point B3 was located as in the Phase II piano, by bisecting the 547 dimension that was used to define the tail/spine point, and raising a perpendicular at that midpoint to intersect the bentside line and define point B3. The rear dimension in this construction is 273 mm versus 287 mm, which partially accounts for the slightly different tail geometry of Phase III versus Phase II Stein pianos.

The front edge of the instrument was located on the spine by first placing the trammel between the spine reference point and the 520 mark on the spine. An arc of radius 520 mm is swung around to meet the spine, thereby defining the front edge of the keywell.

The overall length of the bottom boards of a Phase III Stein can be calculated as 547 + 1040 + 520 = 2107 mm.

## *Discussion*

It cannot be emphasized enough that the constructive geometric procedures described above could be performed without the need for a single measurement, other than that of the modular unit W which is necessary to fix the absolute size of the piano. No knowledge of any other dimension was required, nor would that knowledge have been useful to the builder. It is the proportional relationships between longitudinal and transverse dimensions that are the key to the design methodology. The dimensional values given above are only cited for the purpose of analysis, and for unambiguous written description of the techniques for modern readers who are likely not used to thinking geometrically[28].

The curves of the tail and the treble bentside require (at least) one additional point, besides the two endpoints already constructed. A thin wooden spline bent around these three constructed points would have quite adequately defined consistent bentside and tail curves. It is difficult to re-construct, from an extant instrument, the extra points which were used for these curves. Any proposed solution will probably remain speculative, unless builder's marks can be located on the bottom boards. One pragmatic possibility which is consistent with the observed tail curves is suggested in Figures 6 and 7. The right angle that defines the two tail endpoints is bisected, and the extra tail curve point shown is defined by where this bisector meets the *inside* edge of the bentside liner. This method would necessarily produce slightly different tail curves on the Stein Phase II and III pianos, because the tail dimensions are different (287 vs 273 mm), and, more significantly, because the width of the bentside liner is less on the Phase III piano. To complete the discussion of the tail geometry, it should be noted that the eventual thickness of the (carved down) frame member affects nothing visible, nor does it affect the shape of the bottom boards in the tail, because the tail curve is completely defined by its two endpoints and the extra point.

The predicted length of 2107 mm, derived from the proposed geometry for the bottom boards of a Stein Phase III piano, with W = 465 mm, agrees precisely with the observed measurement taken on the Stein piano in the Wuerttembergisches Landesmuseum, Stuttgart (2107 mm). Note that it is important to record the length of the bottom boards in such instruments, excluding outer case, moldings and any lips that may be added on later. This dimension is not typically reported in the organological literature. A length specified as "excluding lids and moldings"[29] is, in general, different from that of the bottom boards, which cannot be determined without information on the front and tail construction employed by the builder. The length of the 1783 Phase II Stein in the Boston Museum of Fine Arts is reported [30] to be 2110 mm "excluding moldings". This length includes a front lip which the authors have measured to be 30 mm wide on the Stuttgart piano.

---

[28] Modern technical education invariably reduces geometry to its coordinate basis, thereby eliminating most of the practical utility of constructive geometry. This coordinatization of geometry has proceeded at an ever-increasing rate since the advent of computer technology, which is incompatible generally with geometric thinking.

[29] For example Koster Op. Cit.

[30] Latcham and Koster agree on this dimension, although the latter may be the source for the former.

Therefore, using this lip dimension, and assuming the front construction is the same (which is likely), the bottom boards of the Boston Phase II Stein can be estimated to be about 2110 – 30 = 2080 mm. This dimension is consistent with that predicted from the proposed geometry for a Phase II Stein piano.

Further confirmation for the use of constructive geometry as described above is obtained by considering parallel builders, who used the same, or similar schemes, but, working in different locations, began with different half-inch units, and consequently different modular dimensions. For instance a piano by the Tyrolean J. Langerer, essentially a Stein design, began with a considerably larger half-inch unit. The predicted length and width are proportionally larger, and consistent with the dimensions expected from the proposed construction. Such a difference in absolute dimensions, yet similar proportional relationships, can also be reported for a ca 1790 five-octave piano by J.D. Schiedmayer , who used another different half-inch measure.

To demonstrate the agreement between theory and the observed geometry of extant instruments, Figure 8 shows a photograph of the underside of a soundboard, when it was removed from the 1788 Stein Phase III piano in the Germanisches Nationalmuseum, Nuernberg. The outer edge of the inner case follows the outside edge of the soundboard. Superimposed on this photograph is an image of the proposed Stein Phase III geometry, which we derived from measurements of the 1783 Stuttgart Phase III Stein. The proportions of these images have not been altered – aspect ratios have been carefully maintained in the superimposition. The extremely close fit between proposed and observed geometry is clear.

The geometric techniques presented here provide a very simple means to fix the absolute size, and lay out the case shape, of any stringed keyboard instrument. Since geometric design constructions would undoubtedly have been passed on verbally from master to apprentice, there is likely to be no written or published record of the specific constructions used. Furthermore, physical “storage” of the design specifics, in the form of a drawing or diapason, is not required, nor is it necessary to physically “retrieve” information on the design. These problems are circumvented by simply laying out the geometry on the bottom boards, each time a new instrument was begun. This article demonstrates the difficulties involved with succinctly, and accurately, describing the constructions in written form, to a modern audience. Also, the apparent complexity of the descriptions is further increased due to including explanation of the theoretical background behind the “instructions”. The early instrument maker would simply have used the instructions like a recipe, with no need for understanding their reasoning, beginning at the starting point and constructing all the required points one after the other. The entire operation of laying out a five-octave piano can be done in only ten minutes using geometric construction, producing consistently accurate and predictable results, and a piano case into which the stringband will automatically fit.

Figure 1. Simple geometric construction for the golden mean $\phi$ the ratio of DB:CB.

Figure 2. Interlocked golden rectangles and relation to the rectangle of sides $\sqrt{5}$ by 1. This provides the important shop construction for the obtaining the golden mean on a rectangular structure.

Figure 3. Construction of the golden triangle and $\sqrt{\phi}$ proportion.

Figure 4. The rectangular skeleton structure of a five-octave Viennese piano.

Figure 5. Basic shop construction for skeleton structure of (most) five-octave Viennese pianos.

Figure 6. Outline structure and shop construction of 1783 Phase II Stein piano (Museum of Fine Arts, Boston).

Figure 7. Outline structure and shop construction of a 1783 Phase III Stein piano (Wuerttembergisches Landesmuseum, Stuttgart).

Figure 8. Proposed geometric construction derived from 1783 Phase III Stein piano (Wuerttembergisches Landesmuseum, Stuttgart), superimposed on photograph of the underside of a 1788 Phase III Stein soundboard (Germanisches Nationalmuseum, Nuernberg).

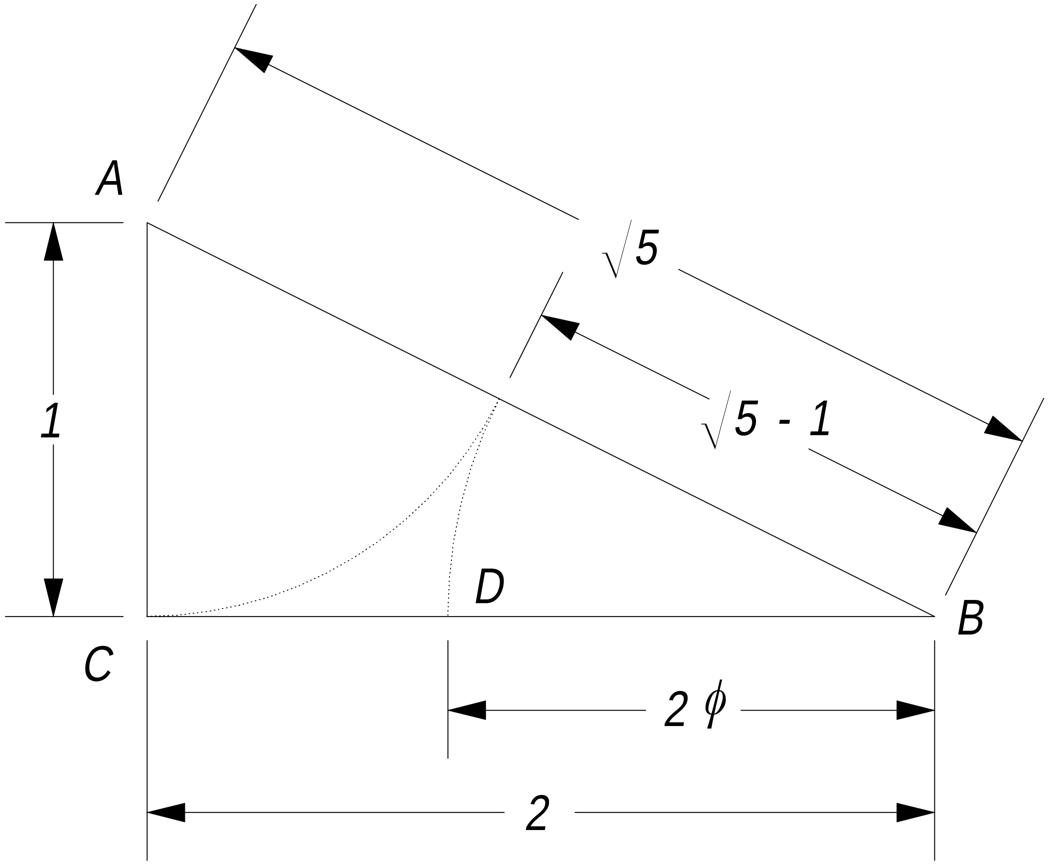

**Figure 1**

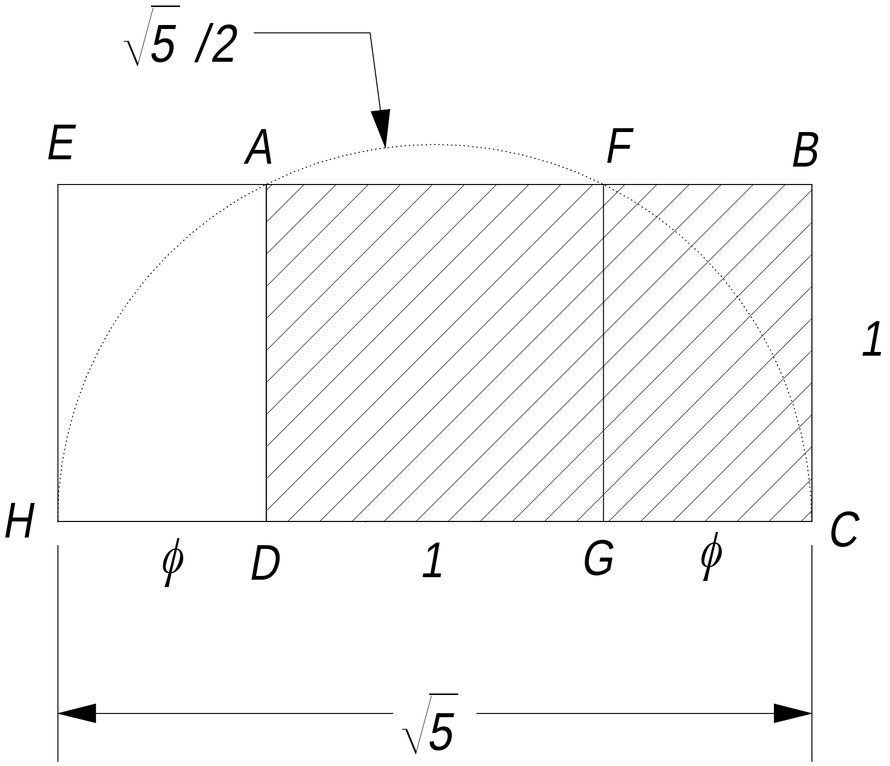

**Figure 2**

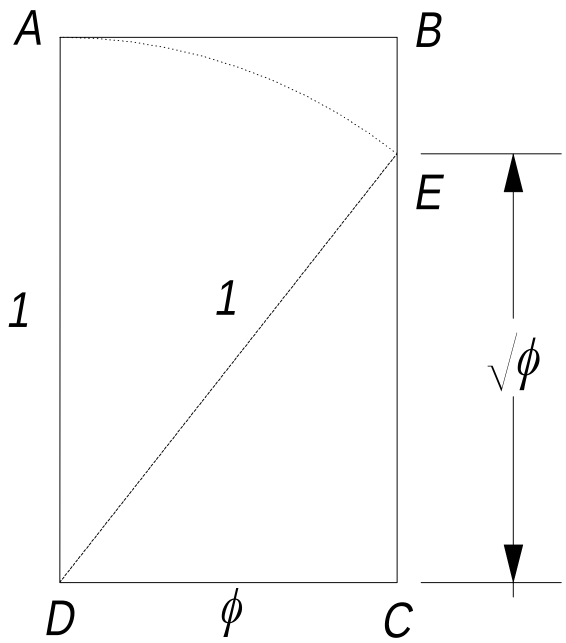

**Figure 3**

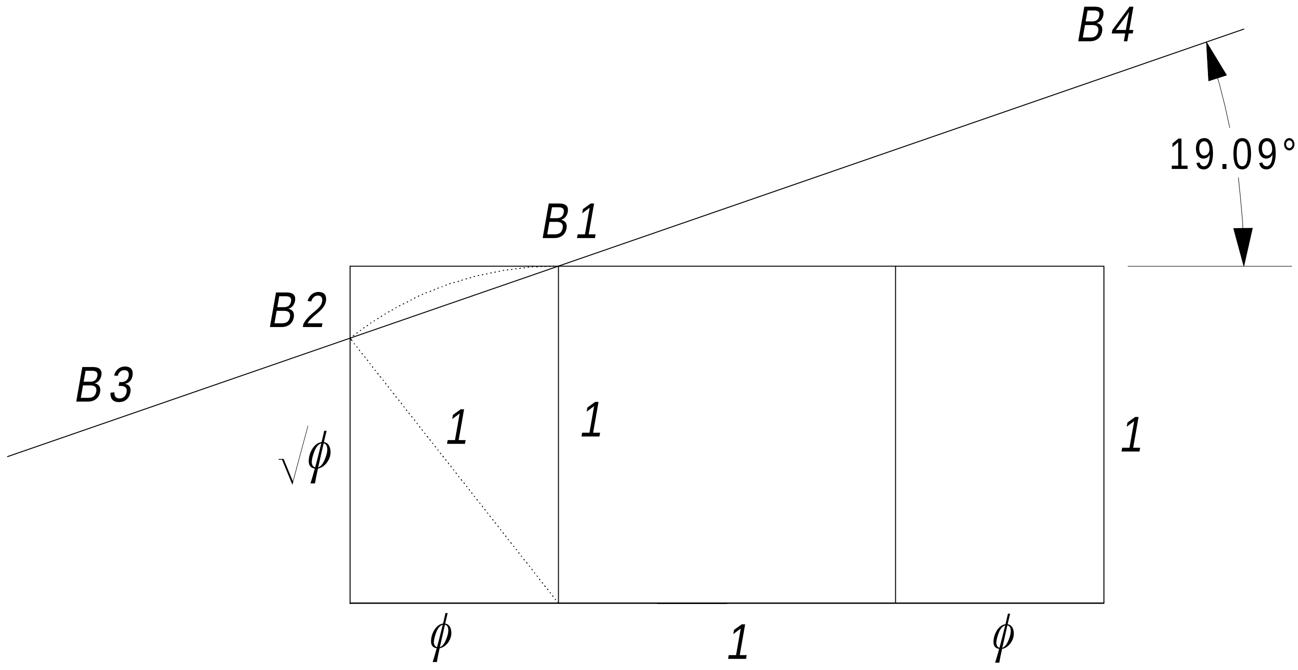

Figure 4

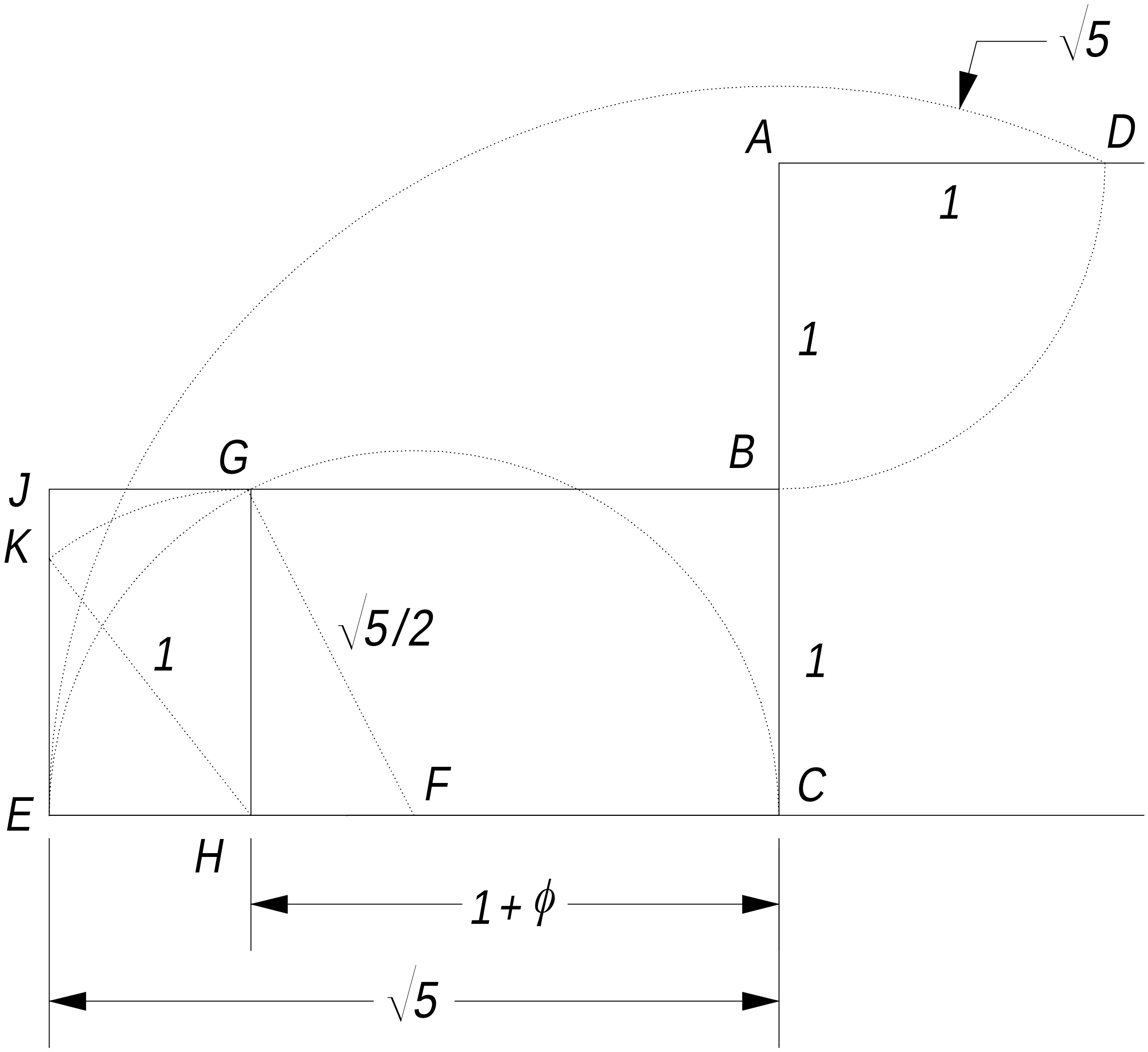

Figure 5

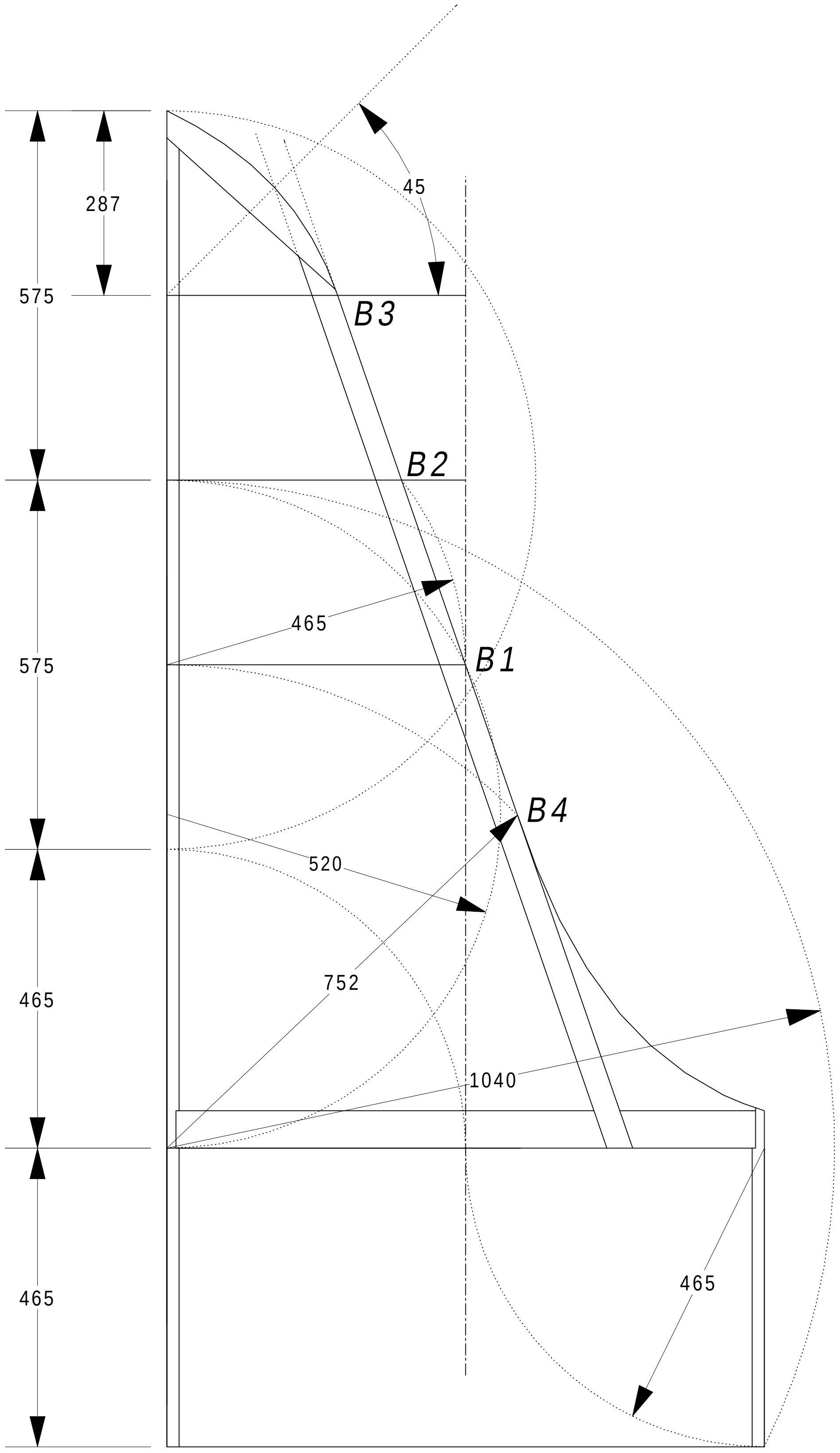

Figure 6

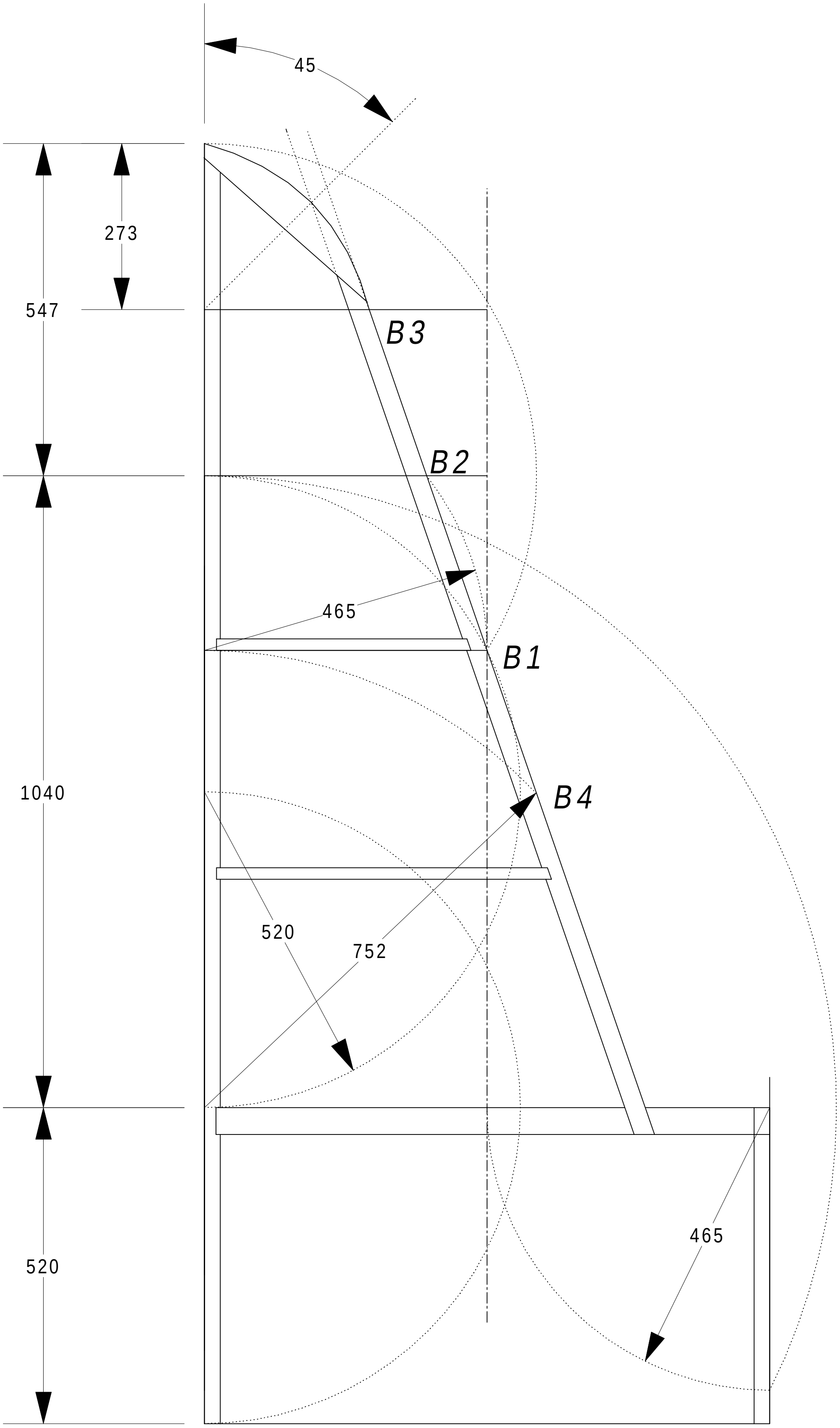

Figure 7

FIGURE 8